The $K(\pi, 1)$ conjecture for affine Artin groups

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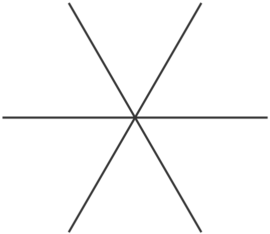
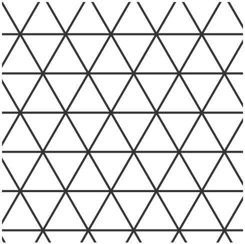
Reflection groups

A *reflection group* is a discrete group generated by orthogonal reflections in a Euclidean space $\mathbb{R}^n$.

To every reflection group W is associated a *hyperplane arrangement*: the set of hyperplanes H such that the reflection with respect to H is an element of W.



Arrangement of a *finite* reflection group

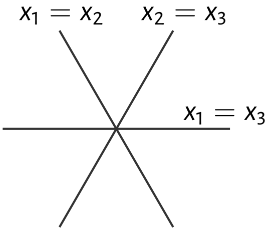


Arrangement of an *infinite* (*affine*) reflection group

The symmetric group $\mathfrak{S}_n$

$\mathfrak{S}_n$ is the group generated by the reflections w.r.t. the hyperplanes $\{x_i = x_j\}$ in $\mathbb{R}^n$: these correspond to the transpositions $(i\ j)$.

Example: $\mathfrak{S}_3$



The arrangement of $\mathfrak{S}_3$ in $\{x_1 + x_2 + x_3 = 0\} \subseteq \mathbb{R}^3$

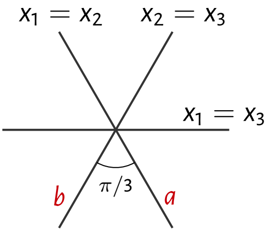
Coxeter groups

A *Coxeter group* is a group presented as follows:

$$W = \langle\, S \mid s^2 = 1 \;\; \forall s \in S,\;\; \underbrace{sts\cdots}_{m_{s,t}\text{ factors}} \;=\; \underbrace{tst\cdots}_{m_{s,t}\text{ factors}} \;\; \forall s \neq t \,\rangle.$$

Reflection groups are particular instances of Coxeter groups: the set S is given by the reflections with respect to the walls of a *fundamental chamber*; the angle between two walls is $\frac{\pi}{m_{s,t}}$.

Example: $\mathfrak{S}_3$



The reflections a and b yield the following Coxeter presentation:

$$\mathfrak{S}_3 = \langle a, b \mid a^2 = b^2 = 1,\; aba = bab \rangle$$

Artin groups

Removing the relations $s^2 = 1$ in the presentation of a Coxeter group, we obtain the corresponding *Artin group*:

$$G_W = \langle\, S \mid \underbrace{sts\cdots}_{m_{s,t}\text{ factors}} = \underbrace{tst\cdots}_{m_{s,t}\text{ factors}} \quad \forall\, s \neq t \,\rangle.$$

Example: the braid group on 3 strands

If $W = \mathfrak{S}_3$ is the symmetric group on 3 letters, the corresponding Artin group is

$$\mathcal{B}_3 = \langle a, b \mid aba = bab \rangle.$$

Other examples

- Free groups (all $m_{s,t} = \infty$)
- Free abelian groups (all $m_{s,t} = 2$)
- Right-angled Artin groups (all $m_{s,t} \in \{2, \infty\}$)

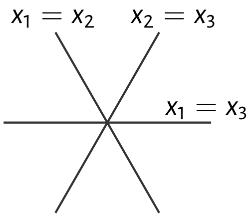
Artin groups (2)

Topologically, an Artin group G_W is the fundamental group of the *configuration space*

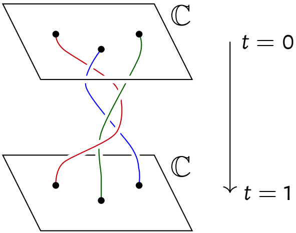
$$Y_W = \left(\mathbb{C}^n \setminus \bigcup_{H \in \mathcal{A}_W} H_{\mathbb{C}} \right) / W.$$

Example (continued): the braid group on 3 strands

$$Y_W = \{(x_1, x_2, x_3) \in \mathbb{C}^3 \mid x_i \neq x_j\} / \mathfrak{S}_3$$



The (real) arrangement



Loops in Y_W are "braids"

Open problems on Artin groups

(1) Artin groups are torsion-free.

(2) Determine the center.

(3) Solve the word problem.

(4) $K(\pi, 1)$ conjecture (Brieskorn-Arnol'd-Pham-Thom '60s):
the configuration space Y_W is a classifying space for G_W.

$\pi_1(Y_W) = G_W$, and the higher homotopy groups are trivial (equivalently, the universal cover of Y_W is contractible).

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Solved for *spherical* Artin groups (Brieskorn-Saito 1971, Deligne 1972).

(1)-(3) solved for *affine* Artin groups (McCammond-Sulway 2017).

(4) solved for some affine Artin groups (Okonek 1979, Callegaro-Moroni-Salvetti 2010).

Theorem (P.-Salvetti 2019)

The $K(\pi, 1)$ conjecture holds for all affine Artin groups.

Interval groups

G group, R generating set, $g \in G$.

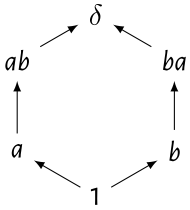
Let $[1, g]^G$ be the interval between 1 and g in the (right) Cayley graph of G.

Definition (Interval group)

Let G_g be the group generated by R, with the relations visible in $[1, g]^G$.

Example

If $G = W$ (a finite Coxeter group), $R = S$, and $g = \delta$ (the longest element), then G_g is the spherical Artin group G_W.



$$W = \langle a, b \mid a^2 = b^2 = 1,\ aba = bab \rangle$$
$$\delta = aba = bab$$

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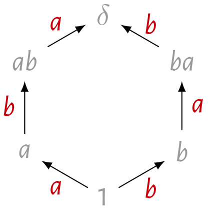


$$W = \langle a, b \mid a^2 = b^2 = 1,\ aba = bab \rangle$$
$$\delta = aba = bab$$

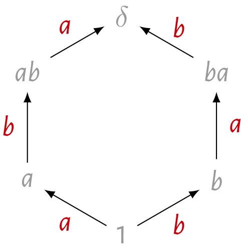
$$W_\delta = \langle a, b \mid aba = bab \rangle$$

Garside groups

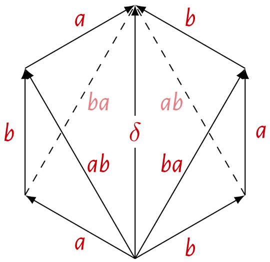
Theorem (Garside 1969, Dehornoy 2002, Bessis 2003)

If $[1, g]^G$ is a *balanced lattice*, then G_g is a *Garside group*, and:

- the elements of G_g have a normal form $g^m x_1 \cdots x_k$, with $x_i \in [1, g]^G$;
- the complex $K_G = \Delta([1, g]^G)/\text{labeling}$ is a classifying space for G_g.



The balanced lattice $[1, \delta]^W$



The associated classifying space K_W

Spherical Artin groups as Garside groups

There are two natural ways to realize spherical Artin groups as Garside groups, from finite Coxeter groups.

Standard Garside structure		Dual Garside structure	
$R = S$	(simple system)	$R = \{\text{all reflections}\}$	
$g = \delta$	(longest element)	$g = w$	(Coxeter element)

Spherical Artin groups as Garside groups

There are two natural ways to realize spherical Artin groups as Garside groups, from finite Coxeter groups.

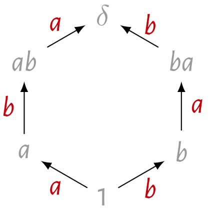
We show the $\mathfrak{S}_3$ example: $W = \langle a, b \mid a^2 = b^2 = 1,\ aba = bab \rangle$.

Standard Garside structure

$R = S = \{a, b\}$ (simple system)

$g = \delta = aba$ (longest element)

$W_\delta = \langle a, b \mid aba = bab \rangle = G_W$



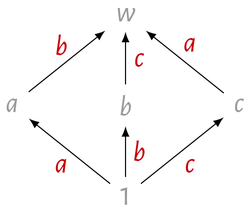
(weak Bruhat order)

Dual Garside structure

$R = \{\text{all reflections}\} = \{a, b, c\}$

$g = w = ab$ (Coxeter element)

$W_w = \langle a, b, c \mid ab = bc = ca \rangle \cong G_W$



(noncrossing partition lattice)

The lattice property

Theorem (Bessis 2003, Brady-Watt 2008)

If W is a *finite* Coxeter group, the associated noncrossing partition poset is a lattice.

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If W is a *finite* Coxeter group, the associated noncrossing partition poset is a lattice.

Theorem (McCammond 2015)

If W is an *affine* Coxeter group, the associated noncrossing partition poset is not a lattice in general.

Proof of the $K(\pi, 1)$ conjecture for affine Artin groups

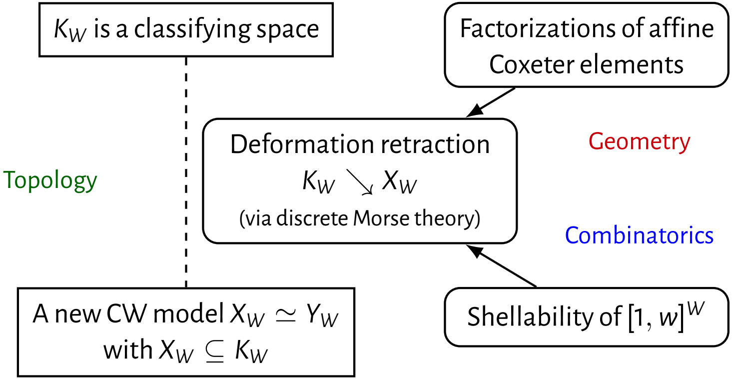
K_W is a classifying space

Proof of the $K(\pi, 1)$ conjecture for affine Artin groups

K_W is a classifying space

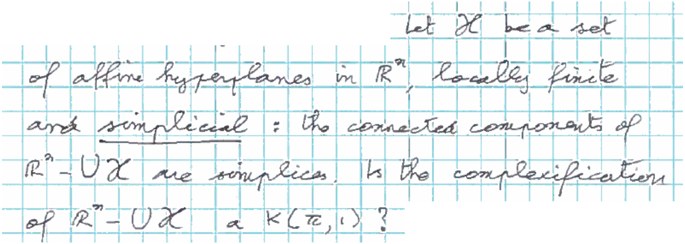
A new CW model $X_W \simeq Y_W$ with $X_W \subseteq K_W$

Proof of the $K(\pi, 1)$ conjecture for affine Artin groups



The road goes ever on

- Arbitrary Coxeter groups (or other families, e.g. hyperbolic)
  - Is K_W a classifying space?
  - What can we say about the factorizations of Coxeter elements?
  - Is $[1, w]^W$ shellable?
  - Does K_W deformation retract onto X_W?
- Affine complex reflection groups
- Simplicial arrangements of affine (real) hyperplanes


Let $\mathcal{H}$ be a set of affine hyperplanes in $\mathbb{R}^n$, locally finite and simplicial: the connected components of $\mathbb{R}^n - \bigcup\mathcal{H}$ are simplices. Is the complexification of $\mathbb{R}^n - \bigcup\mathcal{H}$ a $K(\pi,1)$?

Thanks!

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G. Paolini and M. Salvetti, *Proof of the* $K(\pi, 1)$ *conjecture for affine Artin groups*, arXiv preprint 1907.11795 (2019)