# On a quadratic polynomial attached to simple Lie algebras

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A finite dimensional simple Lie superalgebra $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ is called *basic* if

1. $\mathfrak{g}_{\bar{0}}$ is a reductive Lie algebra;
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### Classification (Kac)

A basic Lie superalgebra is either a simple Lie algebra or it is one of the Lie superalgebras $sl(m|n)$ $(m, n \geq 1, m \neq n)$, $psl(m|m)$ $(m \geq 2)$, $osp(m|n) = spo(n|m)$ $(m \geq 1, n \geq 2 \text{ even})$, $D(2, 1; a)$ $(a \in \mathbb{C},\ a \neq 0, -1)$, $F(4)$, $G(3)$.

# The polynomial $p_{\mathfrak{g}}(k)$

| $\mathfrak{g}$ | $p(k)$ | $\mathfrak{g}$ | $p(k)$ |
|---|---|---|---|
| $sl(m\|n),\ n \neq m$ | $(k+1)(k+(m-n)/2)$ | $E_6$ | $(k+3)(k+4)$ |
| $psl(m\|m)$ | $k(k+1)$ | $E_7$ | $(k+4)(k+6)$ |
| $osp(m\|n)$ | $(k+2)(k+(m-n-4)/2)$ | $E_8$ | $(k+6)(k+10)$ |
| $spo(n\|m)$ | $(k+1/2)(k+(n-m+4)/4)$ | $F_4$ | $(k+5/2)(k+3)$ |
| $D(2,1;a)$ | $(k-a)(k+1+a)$ | $G_2$ | $(k+4/3)(k+5/3)$ |
| $F(4),\ \mathfrak{g}^\natural = so(7)$ | $(k+2/3)(k-2/3)$ | $G(3),\ \mathfrak{g}^\natural = G_2$ | $(k-1/2)(k+3/4)$ |
| $F(4),\ \mathfrak{g}^\natural = D(2,1;2)$ | $(k+3/2)(k+1)$ | $G(3),\ \mathfrak{g}^\natural = osp(3\|2)$ | $(k+2/3)(k+4/3)$ |

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### Drinfeld's Yangian

$$Y(\mathfrak{g})$$

associated to a simple Lie algebra $\mathfrak{g}$.

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In all these problems $p_{\mathfrak{g}}(k)$ will play a key role.

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## References

1. KMP, Communications in Contemporary Mathematics, in press, available online
2. long series of papers with AKMP, all already published
3. KMP, arXiv:2008.13178 + paper in preparation

# Affine Lie superalgebras. Notation

$\mathfrak{g}$ simple basic Lie superalgebra

$$\widehat{\mathfrak{g}} = \mathbb{C}[t, t^{-1}] \otimes \mathfrak{g} \oplus \mathbb{C}K \oplus \mathbb{C}d$$

$K$ is central, $d$ acts as the Euler operator $t\,\frac{d}{dt}$. Set

$$\widehat{\mathfrak{g}}' = [\widehat{\mathfrak{g}}, \widehat{\mathfrak{g}}].$$

If we set $x_{(m)} = t^m \otimes x$, $x \in \mathfrak{g}$, the bracket in $\widehat{\mathfrak{g}}'$ and hence in $\widehat{\mathfrak{g}}$ is described by

$$[x_{(m)}, y_{(n)}] = [x, y]_{(m+n)} + m\delta_{m,-n}(x, y)K,$$

A Cartan subalgebra is $\widehat{\mathfrak{h}} = \mathfrak{h} \oplus \mathbb{C}K \oplus \mathbb{C}d$, and $\widehat{\mathfrak{h}}^* = \mathfrak{h}^* \oplus \mathbb{C}\Lambda_0 \oplus \mathbb{C}\delta$.

# Vertex algebras associated to affine Lie superalgebras

Let $k \in \mathbb{C}$. Consider $\mathbb{C}$ as a $\mathfrak{g} \otimes \mathbb{C}[t] \oplus \mathbb{C}K$-module such that $\mathfrak{g} \otimes \mathbb{C}[t]$ acts trivially and $K$ as the scalar $k$.

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We have the induced $\hat{\mathfrak{g}}'$-module

$$V^k(\mathfrak{g}) = U(\hat{\mathfrak{g}}') \otimes_{U(\mathfrak{g}' \otimes \mathbb{C}[t] \oplus \mathbb{C}K)} \mathbb{C}.$$

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(generalized Verma module)
$V^k(\mathfrak{g})$ has a unique maximal proper submodule and we denote by $V_k(\mathfrak{g})$ the corresponding irreducible quotient. Then, $V_k(\mathfrak{g}) \cong L(k\Lambda_0)$ as $\hat{\mathfrak{g}}$-modules ($k\Lambda_0$ is called the vacuum weight).

## An aside on vertex algebras

Borcherds definition: $(V, T, \mathbf{1}, Y)$, where

- $V$ is a vector superspace
- $\mathbf{1} \in V$
- $T \in End(V)$
- there are bilinear products $(a, b) \mapsto a_{(n)}b$, collected into a generating series

$$V \to End(V)[[z^{\pm 1}]], \quad Y(a, z) = \sum_{n \in \mathbb{Z}} a_{(n)} z^{-n-1}.$$

where for any $b \in V$ there exists $n >> 0$ such that $a_{(n)}b = 0$.

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+ **Axioms**: $Y(\mathbf{1}, z) = Id, \quad a_{(-1)}\mathbf{1} = a$, Borcherds identity:

$$\sum_{j=0}^{\infty} \binom{m}{j} (a_{(n+j)}b)_{(m+k-j)}c =$$

$$\sum_{j=0}^{\infty}(-1)^j \binom{n}{j} (a_{(m+n-j)}b)_{(k+j)}c - p(a, b) \sum_{j=0}^{\infty}(-1)^{j+n} \binom{n}{j} (b_{(n+k-j)}a)_{(m+j)}c$$

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$V^k(\mathfrak{g})$ carries the structure of vertex algebra, uniquely determined by

$$Y(x_{(-1)}\mathbf{1}, z) = \sum_{n \in \mathbb{Z}} x_{(n)} z^{-n-1} \text{ where } x_{(n)} = t^n \otimes x.$$

# VOAs

Vertex operator algebras: it is given a special vector $\omega \in V$, with

$$Y(\omega, z) = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$$

- $L_0$ acts diagonally on $V^k(\mathfrak{g})$ with half-integral eigenvalues bounded below .
- $$[L_m, L_n] = (m-n)L_{m+n} + \delta_{m,-n}\frac{m^3-m}{12}c\, Id.$$
- $L_{-1} = T$

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For $k \neq -h^\vee$, a conformal vector is given by **Sugawara construction**

$$\omega = \frac{1}{2(k+h^\vee)} \sum_{i=1}^{\dim \mathfrak{g}} a^i_{(-1)} b^i_{(-1)} \mathbf{1}, \quad \{a^i\}, \{b^i\} \text{dual bases of } \mathfrak{g}$$

# Minimal quantum affine W-algebras

Choose a Cartan subalgebra $\mathfrak{h}$ for $\mathfrak{g}_{\bar{0}}$ and let $\Delta$ be the set of roots. Fix a minimal root $-\theta$ of $\mathfrak{g}$. We may choose root vectors $e_\theta$ and $e_{-\theta}$ such that

$$[e_\theta, e_{-\theta}] = x \in \mathfrak{h}, \qquad [x, e_{\pm\theta}] = \pm e_{\pm\theta}.$$

Due to the minimality of $-\theta$, the eigenspace decomposition of $ad\, x$ defines a *minimal* $\frac{1}{2}\mathbb{Z}$-grading:

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1,$$

where $\mathfrak{g}_{\pm 1} = \mathbb{C} e_{\pm\theta}$.

# Minimal quantum affine W-algebras

Furthermore, one has

$$\mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x, \quad \mathfrak{g}^\natural = \{a \in \mathfrak{g}_0 \mid (a|x) = 0\}.$$

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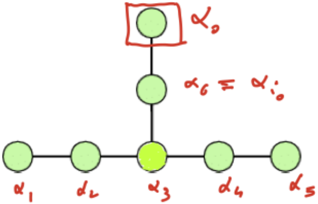
Example: If $\mathfrak{g} = E_6$, then $\mathfrak{g}^\natural$ is of type $A_5$ and $\mathfrak{g}_{1/2} = \Lambda^3\mathbb{C}^6$.



$$E_6 = \mathbb{C}e_\theta \oplus \Lambda^3\mathbb{C}^6 \oplus (sl(6) \oplus \mathbb{C}x) \oplus (\Lambda^3\mathbb{C}^6)^* \oplus \mathbb{C}e_{-\theta}$$

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Note that $\mathfrak{g}^\natural$ is the centralizer of the triple $\{e_{-\theta}, x, e_\theta\}$. We can choose $\mathfrak{h}^\natural = \{h \in \mathfrak{h} \mid (h|x) = 0\}$ as a Cartan subalgebra of the Lie superalgebra $\mathfrak{g}^\natural$, so that $\mathfrak{h} = \mathfrak{h}^\natural \oplus \mathbb{C}x$.
For a given choice of a minimal root $-\theta$, we normalize the invariant bilinear form $(\cdot|\cdot)$ on $\mathfrak{g}$ by the condition

$$(\theta|\theta) = 2. \tag{2.1}$$

The dual Coxeter number $h^\vee$ of the pair $(\mathfrak{g}, \theta)$ is defined to be half the eigenvalue of the Casimir operator of $\mathfrak{g}$ corresponding to $(\cdot|\cdot)$, normalized by (2.1).

# Minimal quantum affine W-algebras

Kac, Wakimoto and Roan associated a vertex algebra $W^k(\mathfrak{g}, f)$, called a *universal W-algebra*, to each triple $(\mathfrak{g}, f, k)$, where $\mathfrak{g}$ is a basic Lie superalgebra, $f$ is a nilpotent element of $\mathfrak{g}_{\bar{0}}$, and $k \in \mathbb{C}$, by applying the quantum Hamiltonian reduction functor $H_f$ to the affine vertex algebra $V^k(\mathfrak{g})$.
In particular, it was shown that, for $k$ non-critical, $W^k(\mathfrak{g}, f)$ has a Virasoro vector $\omega$, making it a conformal vertex algebra, and a set of free generators was constructed. For $k$ non-critical the vertex algebra $W^k(\mathfrak{g}, f)$ has a unique simple quotient, denoted by $W_k(\mathfrak{g}, f)$.
We will consider the case $f = e_{-\theta}$ where $-\theta$ is a minimal root. The Virasoro vector $\omega$ of $W^k_{\min}(\mathfrak{g})$ has central charge

$$c(\mathfrak{g}, k) = \frac{k\, sdim\mathfrak{g}}{k + h^\vee} - 6k + h^\vee - 4.$$

# Interlude: Lie conformal vs vertex algebras

A Lie conformal algebra is a vector space $R$ with a $\mathbb{C}$-linear map

$$[\cdot_\lambda \cdot] : R \otimes R \to \mathbb{C}[\lambda] \otimes R$$

satisfying Lie algebra-like axioms. We have

$$[a_\lambda b] = \sum_{n=0}^{\infty} a_{(n)} b \, \frac{\lambda^n}{n!}$$

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To enhance this structure to a vertex algebra structure one needs a bilinear product $(a, b) \mapsto : ab :$ called *normal order* (+ properties). Setting

$$: ab := a_{(-1)} b$$

one recovers $a_{(n)} b, n < 0$. Examples

$$V^k(\mathfrak{g}) : R = \mathbb{C}[T] \otimes \mathfrak{g} \oplus \mathbb{C}K, \quad [a_\lambda b] = [a, b] + \lambda (a, b) K$$

$$Vir : R = \mathbb{C}[T] \otimes \omega \oplus \mathbb{C}c, \quad [\omega_\lambda \omega] = (T + 2\lambda)\omega + \frac{\lambda^3}{12} c$$

# Presentation of $W^k_{\min}(\mathfrak{g})$

Recall that $\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_{-1/2} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{1/2} \oplus \mathfrak{g}_1, \mathfrak{g}_0 = \mathfrak{g}^\natural \oplus \mathbb{C}x$.

Theorem (KW)

*(a) The vertex algebra $W^k_{\min}(\mathfrak{g})$ is strongly and freely generated by elements $J^{\{a\}}$, where $a$ runs over a basis of $\mathfrak{g}^\natural$, $G^{\{v\}}$, where $v$ runs over a basis of $\mathfrak{g}_{-1/2}$, and the Virasoro vector $\omega$.*

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*Rough explanation: a vector space basis $W^k_{\min}(\mathfrak{g})$ is given by monomials in (iterated) normal orders of the above generators and their derivatives.*

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*(a) The vertex algebra* $W^k_{\min}(\mathfrak{g})$ *is strongly and freely generated by elements* $J^{\{a\}}$*, where* $a$ *runs over a basis of* $\mathfrak{g}^\natural$*,* $G^{\{v\}}$*, where* $v$ *runs over a basis of* $\mathfrak{g}_{-1/2}$*, and the Virasoro vector* $\omega$*.*
*(b) The elements* $J^{\{a\}}$*,* $G^{\{v\}}$ *are primary of conformal weight* $1$ *and* $3/2$*, respectively, with respect to* $\omega$*.*

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*(b) The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight 1 and 3/2, respectively, with respect to $\omega$.*

*Rough explanation: being primary means having a certain $\lambda$-bracket with $\omega$. In this case*

$$[\omega_\lambda J^{\{a\}}] = (L_{-1} + \lambda)J^{\{a\}}, \quad [\omega_\lambda G^{\{v\}}] = (L_{-1} + \tfrac{3}{2}\lambda)G^{\{v\}}$$

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*(b) The elements $J^{\{a\}}$, $G^{\{v\}}$ are primary of conformal weight $1$ and $3/2$, respectively, with respect to $\omega$.*
*(c) The following $\lambda$-brackets hold:*

$$[J^{\{a\}}{}_\lambda J^{\{b\}}] = J^{\{[a,b]\}} + \lambda \left((k + h^\vee/2)(a|b) - \tfrac{1}{4}\kappa_0(a,b)\right),\ a, b \in \mathfrak{g}^\natural,$$

*where $\kappa_0$ is the Killing form of $\mathfrak{g}_0$, and*

$$[J^{\{a\}}{}_\lambda G^{\{u\}}] = G^{\{[a,u]\}},\ a \in \mathfrak{g}^\natural,\ u \in \mathfrak{g}_{-1/2}.$$

*(d) There are explicit formulas yielding $[G^{\{u\}}{}_\lambda G^{\{v\}}]$ for $u, v \in \mathfrak{g}_{-1/2}$.*

# Presentation of $W^k_{\min}(\mathfrak{g})$

**Proposition (AKMPP, J-alg)**

Let $u, v \in \mathfrak{g}_{-1/2}$. Then

$$G^{\{u\}}{}_{(2)}G^{\{v\}} = 4(e_\theta|[u,v])p_\mathfrak{g}(k)\mathbf{1}.$$

Moreover, the linear polynomial $k_i(k)$, $i \in I$, defined by $k_i(k) = k + \frac{1}{2}(h^\vee - h^\vee_{0,i})$, divides $p_\mathfrak{g}(k)$ and

$$G^{\{u\}}{}_{(1)}G^{\{v\}} = 4\sum_{i\in I}\frac{p_\mathfrak{g}(k)}{k_i(k)}J^{\{([[e_\theta,u],v])^\natural_i\}}$$

where $(a)^\natural_i$ denotes the orthogonal projection of $a \in \mathfrak{g}_0$ onto $\mathfrak{g}^\natural_i$.

# Collapsing levels

**Definition**

We say that a level $k$ is *collapsing* if $W_k^{\min}(\mathfrak{g}) = V_{k'}(\mathfrak{g}^\natural)$.

**Theorem (AKMPP)**

*Let* $\mathfrak{g}^\natural = \bigoplus_{i\in I} \mathfrak{g}_i^\natural$. *Then* $k$ *is a collapsing level, if and only if* $p_{\mathfrak{g}}(k) = 0$

$$W_k^{\min}(\mathfrak{g}) = \bigotimes_{i\in I : k_i\neq 0} V_{k_i}(\mathfrak{g}_i^\natural). \tag{2.2}$$

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### Proposition (Arakawa-Moreau for $\mathfrak{g}$ even J.I.M. Jussieu; AKMPP, J-alg)

$W_k^{\min}(\mathfrak{g}) = \mathbb{C}\mathbf{1}$ if and only if either $k = -\frac{h^\vee}{6} - 1$ and either $\mathfrak{g}$ is a Lie algebra belonging to the Deligne's series $(A_2, G_2, D_4, F_4, E_6, E_7, E_8)$ or $\mathfrak{g} = psl(m|m)$ $(m \geq 2)$, $\mathfrak{g} = osp(n+8|n)$ $(n \geq 2)$, $\mathfrak{g}$ is of type $F(4)$, $G(3)$ (for both choices of $\theta$), or $\mathfrak{g} = spo(2|1)$; or $k = -\frac{1}{2}$ and $\mathfrak{g} = spo(n|m)$ $(n > 1, m \neq n+1)$.

# Drinfeld's Yangians

The Yangian $Y(\mathfrak{g})$ of a a simple Lie algebra $\mathfrak{g}$ is the unital associative $\mathbb{C}$-algebra generated by $\{X : X \in \mathfrak{g}\} \cup \{J(X) : X \in \mathfrak{g}\}$ with relations

$$XY - YX = [X, Y]_{\mathfrak{g}}, \quad J([X, Y]) = [J(X), Y], \tag{3.1}$$

$$J(cX + dY) = cJ(X) + dJ(Y), \tag{3.2}$$

$$[J(X), [J(Y), Z]] - [X, [J(Y), J(Z)]] = \tag{3.3}$$

$$\sum_{\lambda,\mu,\nu\in\Lambda} ([X, X_\lambda], [[Y, X_\mu], [Z, X_\nu]])\{X_\lambda, X_\mu, X_\nu\},$$

$$[[J(X), J(Y)], [Z, J(W)]] + [[J(Z), J(W)], [X, J(Y)]] = \tag{3.4}$$

$$\sum_{\lambda,\mu,\nu\in\Lambda} (([X, X_\lambda], [[Y, X_\mu], [[Z, W], X_\nu]])+$$

$$([Z, X_\lambda], [[W, X_\mu], [[X, Y], X_\nu]]))\,\{X_\lambda, X_\mu, J(X_\nu)\}$$

$$\{x_1, x_2, x_3\} = \frac{1}{24} \sum_{\pi\in\mathfrak{S}_3} x_{\pi(1)}x_{\pi(2)}x_{\pi(3)}$$

# Remarks on the definition

It was pointed out by Drinfeld that

1. when $\mathfrak{g} \cong sl_2$ the relation (3.3) follows from (3.1) together with (3.2), and (3.4)
2. when $\mathfrak{g} \not\cong sl_2$ the relation (3.4) follows from the relations (3.1)–(3.3).

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Two remarkable properties of $Y(\mathfrak{g})$:

1. $Y(\mathfrak{g})$ is an Hopf algebra.
2. $Y(\mathfrak{g})$ is a quantization of $U(\mathfrak{g}[t])$: more precisely, if we assign degree 0 to $x \in \mathfrak{g}$ and degree 1 to $J(x)$, $x \in \mathfrak{g}$, the associated graded algebra is isomorphic to $U(\mathfrak{g}[t])$ via the map $x \mapsto x$, $J(x) \mapsto t\,x$.

# Minimal quantization of the adjoint representation

**Theorem (Drinfeld)**

*Let $\mathfrak{g}$ be a simple Lie algebra. Let $\mathcal{V} = \mathfrak{g} \oplus \mathbb{C}$.*

1. *If $\mathfrak{g} = sl(2)$ then for all $\delta \in \mathbb{C}$ the natural action of $\mathfrak{g}$ on $\mathcal{V}$ extends to an action of $Y(\mathfrak{g})$ by setting*

$$J(x)(y, \lambda) = (\delta \lambda x, (x, y)). \tag{3.5}$$

2. *If $\mathfrak{g} \neq sl(2)$ then there exists a unique constant $\delta \in \mathbb{C}$ such that (3.5) extends the natural action of $\mathfrak{g}$ on $\mathcal{V}$ to an action of $Y(\mathfrak{g})$.*
3. *Let $\theta = \sum_i n_i \alpha_i$. Assume that either $n_i = 1$ or $n_i = (\beta, \beta)/(\alpha_i, \alpha_i)$. The fundamental representation $V_{\omega_i}$ of $\mathfrak{g}$ extends to a $Y(\mathfrak{g})$-representation by letting $J(x)$ act as 0.*

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$$p_{\mathfrak{g}}(k) = k^2 + (h^\vee/2 + 1)k + 2\delta.$$

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$$p_{\mathfrak{g}}(k) = k^2 + (h^\vee/2 + 1)k + 2\delta.$$

Moreover $p_{\mathfrak{g}}(k)$ appears in connection with the singularities of the R-matrix for $\mathcal{V}$ (Chari-Pressley)

# A proof of Drinfeld's Theorem

Recall that we want to prove that the natural action of $\mathfrak{g}$ on $\mathcal{V} = \mathfrak{g} \oplus \mathbb{C}$ extends to an action of $Y(\mathfrak{g})$ by setting

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It suffices to check on $(0,1),(U,0)$. Relations (3.1), (3.2) are easy and do not impose any condition on $\delta$.
It is clear that both sides of (3.3), the first Drinfeld's terrific relation, act on $(0,1)$ trivially.

# A proof of Drinfeld's Theorem

The L.H.S of (3.3) $[J(X), [J(Y), Z]] - [X, [J(Y), J(Z)]]$ acts on $(U, 0)$ as

$$([J(X), [J(Y), Z]] - [X, [J(Y), J(Z)]](U, 0) = (f(X, Y, Z)(U), 0)$$

where

$$\begin{aligned} f(X, Y, Z)(U) =& \delta(([Y, Z], U)X - (X, U)[Y, Z] - (Z, U)[X, Y] \\ & + (Y, U)[X, Z] + (Z, [X, U])Y - (Y, [X, U])Z). \end{aligned}$$

Identifying $\mathfrak{g}$ with $\mathfrak{g}^*$ and $End(\mathfrak{g}, \mathfrak{g}) = \mathfrak{g} \otimes \mathfrak{g}^*$ with $\mathfrak{g} \otimes \mathfrak{g}$ via the invariant form we find

$$f(X, Y, Z) = -\delta\partial_3(X \wedge Y \wedge Z)$$

where $\partial_3$ is the usual boundary for Lie algebra homology.

# A proof of Drinfeld's Theorem

We let the R.H.S. of (3.3) act on $(U, 0)$:

$$\sum_{\lambda,\mu,\nu\in\Lambda}([X, x_\lambda], [[Y, x_\mu], [Z, x_\nu]])\{x_\lambda, x_\mu, x_\nu\}(U, 0)$$
$$= (\frac{1}{24}\sum_{\sigma}\sum_{p_1,p_2,p_3}([X, x_{p_1}], [[Y, x_{p_2}], [Z, x_{p_3}]])[x_{p_{\sigma(1)}}, [x_{p_{\sigma(2)}}, [x_{p_{\sigma(3)}}, U], 0)$$
$$= (\frac{1}{24}g_3(X \wedge Y \wedge Z)(U), 0)$$

where $g_3 : \bigwedge^3 \mathfrak{g} \to End(\mathfrak{g})$.

# A proof of Drinfeld's Theorem

Consider the maps $G_2 : \bigwedge^2 \mathfrak{g} \to S^3(\mathfrak{g}^*)$ and $G_3 : \bigwedge^3 \mathfrak{g} \to S^3(\mathfrak{g}^*)$ defined by setting

$$G_2(X \wedge Y)(a) = ([[X, a], a], [Y, a]), \; G_3(X \wedge Y \wedge Z)(a) = ([[X, a], [Y, a]], [Z, a])$$

**Lemma (suggested by C. De Concini)**

*$G_2, G_3$ are $\mathfrak{g}$-equivariant. Moreover*

$$G_3 = \frac{1}{3} G_2 \circ \partial_3$$

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Identify $S^3(\mathfrak{g}^*)$ and $S^3(\mathfrak{g})$ using the form $(\cdot, \cdot)$. Set

$$g_i = ad \circ Symm \circ G_i : \bigwedge^i \mathfrak{g} \to End(\mathfrak{g}) \tag{3.6}$$

where $Symm : S(\mathfrak{g}) \to U(\mathfrak{g})$ is the symmetrization map, $ad$ is the extension to $U(\mathfrak{g})$ of the adjoint representation $ad : \mathfrak{g} \to End(\mathfrak{g})$.

# A proof of Drinfeld's Theorem

## Remark

$\hat{E}g_3$ is exactly the map arising in the action of the RHS of Drinfeld terrific relation. Recall that in the end we want to prove that $g_3$ equals $\partial_3$ up to a constant (unique if $\mathfrak{g}$ is not $sl(2)$).

## Lemma

*Identify* $End(\mathfrak{g})$ *with* $\mathfrak{g} \otimes \mathfrak{g}$ *via* $A \mapsto \sum_i A(x_i) \otimes x^i$. *Then*

$$Im(g_i) \subset \bigwedge^2 \mathfrak{g}.$$

# A proof of Drinfeld's Theorem

## Lemma

*There is a costant* $k \in \mathbb{C}$ *such that*

$$g_3 = k\,\partial_3. \tag{3.7}$$

*If* $\mathfrak{g} = sl(2)$ *then* (3.7) *holds for all* $k \in \mathbb{C}$*; if* $\mathfrak{g} \neq sl(2)$ *then* $k$ *is unique.*

# A proof of Drinfeld's Theorem

### Lemma

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### Corollary

Choosing $\delta = -\frac{k}{24}$ we see that the relation (3.3) is verified, and statements (1) and (2) in Drinfeld's Theorem are proven.

# A proof of Drinfeld's Theorem

### Lemma

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### Remark

The proof uses De Concini's Lemma and the following decompositions by Kostant

$$\bigwedge^2 \mathfrak{g} = \mathbf{d}\mathfrak{g} \oplus U_2, \quad \bigwedge^3 \mathfrak{g} = Ker\,\partial_3 \oplus Im\,\mathbf{d} = Ker\,\partial_3 \oplus \mathbf{d}(U_2)$$

where $\mathbf{d}$ is the Chevalley-Eilenberg differential for Lie algebra cohomology, $U_2$ is the $\mathfrak{g}$-submodule of $\bigwedge^2 \mathfrak{g}$ generated by 2-tensors $x \wedge y$, $[x, y] = 0$.

# A formula for $\delta$

## Proposition

If $\mathfrak{g}^\natural$ splits into $r$ simple components, then

$$\delta = -\frac{12\sum_{s=1}^{r}(h^\vee + 1/2 - \bar{h}_s^\vee)\gamma_s + 45}{144}. \tag{3.8}$$

where $\bar{h}_s^\vee, \gamma_s$ are (explicit) constants depending on the gradation.

## Remark

Relation (3.8) easily gives Drinfeld's formulas for $\delta$:

$$\delta^{Killing} = \begin{cases} -\frac{1}{32n^2} & \text{if } \mathfrak{g} = sl(n), \\ -\frac{n-4}{16(n-2)^3} & \text{if } \mathfrak{g} = so(n), \\ -\frac{n+2}{64(n+1)^3} & \text{if } \mathfrak{g} = sp(2n), \\ -\frac{5}{144(\dim\mathfrak{g}+2)} & \text{if } \mathfrak{g} \text{ is of exceptional type.} \end{cases}$$

# Relationships with minimal $W$-algebras

It is known by Kac-Wakimoto that there is a vertex algebra $W$ generated by fields $L$, $J^v$ with $v \in \mathfrak{g}^\natural$, $G^u$ with $u \in \mathfrak{g}_{-1/2}$ with the following OPE: $L$ is a Virasoro vector with central charge $\frac{k \dim \mathfrak{g}}{k+h^\vee} - 6k + h^\vee - 4$, $J^u$ are primary of conformal weight 1, $G^v$ are primary of conformal weight $\frac{3}{2}$ and

1. $[J^v{}_\lambda J^w] = J^{[v,w]} + \lambda\delta_{ij}(k + \frac{h^\vee - h_i^\vee/\nu_i}{2})(v|w)$ for $v \in \mathfrak{g}_i^\natural$, $w \in \mathfrak{g}_j^\natural$;
2. $[J^v{}_\lambda G^u] = G^{[v,u]}$ for $u \in \mathfrak{g}_{-1/2}$, $v \in \mathfrak{g}^\natural$;
3. $[G^u{}_\lambda G^v] = a(u,v,k) + \lambda b(u,v,k) + \frac{\lambda^2}{2} c(u,v,k)$ for $u, v \in \mathfrak{g}_{-1/2}$ with $c(u,v,k) \in \mathbb{C}$, $\Delta(b(u,v,k)) = 1$, and $\Delta(a(u,v,k)) = 2$.

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## Basic Idea

We want to investigate the conditions on $a, b, c$ imposed by the vertex algebra axioms.

# Relationships with minimal $W$-algebras

**Proposition**

We can normalize $[G^v{}_\lambda G^w]$ in such a way that, setting

$$p(k) = \begin{cases} (k + \frac{h^\vee - \bar{h}_1^\vee}{2})(k + \frac{h^\vee - \bar{h}_2^\vee}{2}) & \text{if } \mathfrak{g}^\natural \text{ has two components,} \\ (k + \frac{h^\vee - \bar{h}_1^\vee}{2})(k + \frac{\bar{h}_1^\vee}{2} + 1) & \text{otherwise.} \end{cases}$$

1. $c(u, v, k) = 4p(k)(e_\theta, [u, v])$
2. the expression for $[G^v{}_\lambda G^w]$ coincides with the one given by Kac and Wakimoto.

# The concidence

## Theorem

*If* $\mathfrak{g}^\natural$ *has one or three components, then*

$$\delta = -\frac{1}{2}\left(\frac{h^\vee - \bar{h}_1^\vee}{2}\right)\left(\frac{\bar{h}_1^\vee}{2} + 1\right),$$

*while if* $\mathfrak{g}^\natural$ *has two components, then*

$$\delta = -\frac{1}{2}\left(\frac{h^\vee - \bar{h}_1^\vee}{2}\right)\left(\frac{h^\vee - \bar{h}_2^\vee}{2}\right)$$

*and* $\bar{h}_1^\vee + \bar{h}_2^\vee = h^\vee - 2$. *In particular, in both cases,*

$$p_{\mathfrak{g}}(k) = k^2 + \left(\frac{h^\vee}{2} + 1\right)k - 2\delta.$$

# Some remarkable identities

## Proposition

If $\mathfrak{g}^\natural$ has two components, then

$$\dim\mathfrak{g} = \frac{(h^\vee+1)\left(2(h^\vee)^2 + h^\vee(\bar{h}_1^\vee - 2) - \bar{h}_1^\vee(\bar{h}_1^\vee+2)\right)}{(\bar{h}_1^\vee+2)(h^\vee - \bar{h}_1^\vee)}, \quad \bar{h}_1^\vee + \bar{h}_2^\vee = h^\vee - 2.$$

Otherwise

$$\dim\mathfrak{g} = \frac{2\left(5(h^\vee)^2 - h^\vee - 6\right)}{h^\vee+6}, \quad \bar{h}_1^\vee = \frac{2(h^\vee-3)}{3}$$

or

$$\dim\mathfrak{g} = 2(h^\vee)^2 - 3h^\vee + 1, \quad \bar{h}_1^\vee = h^\vee - 1. \tag{4.1}$$

Moreover, (4.1) occurs if and only if $\mathfrak{g}^\natural$ is simple and $\alpha_{i_0}$ is short.

# Unitarity

Let $V$ be a conformal vertex algebra with $\omega = \sum_{n\in\mathbb{Z}} L_n z^{-n-2}$
Let $\psi$ be a conjugate linear involution of $V$.

### Definition

A Hermitian form $H(\cdot\,,\cdot)$ on $V$ is called $\psi$–*invariant* if, for all $a \in V$, one has

$$H(v, Y(a,z)u) = H(Y(A(z)a, z^{-1})v, u), \quad u, v \in V.$$

Here the linear map $A(z) : V \to V((z))$ is defined by

$$A(z) = e^{zL_1} z^{-2L_0} g,$$

where

$$g(a) = e^{-\pi\sqrt{-1}(\frac{1}{2}p(a)+\Delta_a)}\psi(a), \quad a \in V,$$

$p(a) = 0$ or $1$ in $\mathbb{Z}$ stands for the parity of $a$, and $\Delta_a$ stands for its $L_0$–eigenvalue.

# Temptative explanation of the definition

Consider the easier case of symmetric forms on VOAS (e.g. $V^k(\mathfrak{g}), \mathfrak{g}$ even).

- Frenkel-Lepowsky-Meurman The module structure on the dual $M' = \bigoplus_i M_i^*$ of a module $M = \bigoplus_i M_i$ is given by
$$\langle Y(v,z)m', m\rangle = \langle m', Y(e^{zL_1}(-z^{-2})^{L_0}v, z^{-1})m\rangle$$
$v \in V, m \in M, m' \in M'$
- Haisheng Li The space of bilinear invariant forms on a vertex algebra is isomorphic to the linear dual of $V_0/L_1V_1$
- in this simplified setting, our $g$ reduces to
$$g = (-1)^{L_0}$$
so the invariance on the form amouts to
$$\begin{aligned} H(v, Y(a,z)u) &= H(Y(e^{zL_1}z^{-2L_0}(-1)^{L_0}a, z^{-1})v, u), \\ &= H(Y(e^{zL_1}(-z^{-2})^{L_0}a, z^{-1})v, u), \quad u, v \in V. \end{aligned}$$

# Unitarity

**Definition**

We say that a conformal vertex algebra $V$ is *unitary* if there exists a conjugate linear involution $\psi$ of $V$ and a $\psi$-invariant positive definite Hermitian form on $V$.

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## Example

It is a theorem of Kac (holding more generally for representations of affine algebras) that $V_k(\mathfrak{g})$ is unitary if and only $\psi$ is the compact involution of $\mathfrak{g}$ and $k$ is a non-negative integer.

# Superconformal algebras

Recall that all well-known superconformal algebras in conformal field theory are the minimal $W$-algebras or are obtained from them by a simple modification:

(a). $W_k^{\min}(spo(2|N))$ is the Virasoro vertex algebra for $N = 0$, the Neveu-Schwarz vertex algebra for $N = 1$, the $N = 2$ vertex algebra for $N = 2$, and becomes the $N = 3$ vertex algebra after tensoring with one fermion; it is the Bershadsky.Knizhnik algebra for $N > 3$;

(b). $W_k^{\min}(psl(2|2))$ is the $N = 4$ vertex algebra;

(c). $W_k^{\min}(D(2, 1; a))$ tensored with four fermions and one boson in the big $N = 4$ vertex algebra;

(d). $W_k^{\min}(F(4))$ and $W_k^{\min}(G(3))$ are the Shatashvili-Vafa vertex algebras.

# Superconformal algebras: unitarity

The unitary Virasoro ($N = 0$), Neveu-Schwarz ($N = 1$) and $N = 2$ simple vertex algebras were classified in the mid 80s. Up to isomorphism, these vertex algebras depend only on the central charge $c(k)$; putting $k = \frac{1}{p} - 1$ we have

$$c(k) = 1 - \frac{6}{p(p+1)} \quad \text{for Virasoro vertex algebra,} \tag{5.1}$$

$$c(k) = \frac{3}{2}\left(1 - \frac{8}{p(p+2)}\right) \quad \text{for Neveu-Schwarz vertex algebra,} \tag{5.2}$$

$$c(k) = 3\left(1 - \frac{2}{p}\right) \quad \text{for } N = 2 \text{ vertex algebra.} \tag{5.3}$$

**Theorem**

*The complete list of unitary $N = 0, 1,$ and $2$ vertex algebras is as follows: either $c(k)$ is given by (5.1), (5.2) or (5.3) respectively, for $p \in \mathbb{Z}_{\geq 2}$ or $c(k) \geq 1, \frac{3}{2}, 3$ respectively.*

# Unitarity of minimal $W$-algebras

$$sl(2|m) \text{ for } m \geq 3, \quad psl(2|2), \quad spo(2|m) \text{ for } m \geq 0,$$
$$osp(4|m) \text{ for } m > 2 \text{ even}, \quad D(2,1;a), \text{ for } a \in R, \quad F(4), \quad G(3).$$

### Theorem

*Let $\mathfrak{g}$ be a Lie superalgebra in the above list $\mathfrak{g}^\natural$ non abelian and let $W_k^{\min}(\mathfrak{g})$ be the corresponding minimal simple $W$-algebra. Assume that $k \neq -h^\vee$. Then $W_k^{\min}(\mathfrak{g})$ is unitary precisely in the following cases:*

1. *$\mathfrak{g}^\natural$ is semisimple and $z_i(k) \in \mathbb{Z}_+$ for all $i > 0$.*
2. *$\mathfrak{g} = sl(2|m), m \geq 3, \ k = -1$.*

*The $z_i$ are explicit linear polynomials in $k$ in bijections with the simple components of $\mathfrak{g}^\natural$.*

# Unitarity of minimal $W$-algebras

## Theorem

*Let $\mathfrak{g}$ be a basic simple Lie superalgebra as above. Let $V = W^{\min}_{-k}(\mathfrak{g})$ with $k \neq h^\vee$. Assume that $\mathfrak{g} \neq spo(2|m), m = 0, 1, 2$, which correspond to the well-understood cases of $V$ Virasoro, Neveu-Schwarz and $N = 2$ superconformal algebra. Then $V$ is a non-trivial unitary vertex algebra if and only if*

1. $\mathfrak{g} = sl(2|m), m \geq 3, \ k = 1$;
2. $\mathfrak{g} = psl(2|2), \ k \in \mathbb{N} + 1$;
3. $\mathfrak{g} = spo(2|3), \ k \in \frac{1}{4}(\mathbb{N} + 2)$;
4. $\mathfrak{g} = spo(2|m), m > 3, \ k = \frac{1}{2}(\mathbb{N} + 1)$;
5. $\mathfrak{g} = D(2, 1; -\frac{m}{m+n}), \ k = \frac{mn}{m+n}, m, n \in \mathbb{N}$;
6. $\mathfrak{g} = F(4), \ k \in \frac{2}{3}(\mathbb{N} + 1)$;
7. $\mathfrak{g} = G(3), \ k \in \frac{3}{4}(\mathbb{N} + 1)$.

# Final remarks

1. The minimal $W$-algebras corresponding to $\mathfrak{g}$ *not* in the above list are almost never unitary: the precise result is given in our arXiv paper.
2. **Relation with $p_{\mathfrak{g}}(k)$.** Write $\approx$ to mean equal up to a constant factor. Then

$$p_{\mathfrak{g}}(k) \approx \begin{cases} z_1(k) z_2(k) & \text{if } \mathfrak{g}^\natural \text{ has two components,} \\ z_1(k)(k + \frac{\bar{h}_1^\vee}{2} + 1) & \text{otherwise.} \end{cases}$$