Parabolic restriction for Coulomb branch algebras and categorical g-actions for truncated shifted Yangians

Joel Kamnitzer

University of Toronto

April 27, 2020

Geometric Satake correspondence

- G, simply-laced, semisimple, simply-connected complex group
- P weight lattice, P_+ dominant weights
- $\lambda \in P_+$, $V(\lambda)$ irreducible representation of highest weight λ
- $\mu \in P$, $V(\lambda)_\mu$ is the μ weight space

Geometric incarnation from the affine Grassmannian

- $Gr = G^\vee((t))/G^\vee[[t]]$, the affine Grassmannian of the Langlands dual group
- For $\mu \in P$, a point $t^\mu \in Gr$
- For $\lambda \in P_+$, $Gr^\lambda = G^\vee[[t]]t^\lambda$, $\quad \overline{Gr^\lambda} = \cup_{\mu \leq \lambda} Gr^\mu$
- For $\mu \in P$, $S^\mu = N_-^\vee((t))t^\mu$

Theorem (Mirkovic-Vilonen)

$H_{top}(\overline{Gr^\lambda} \cap S^\mu) \cong V(\lambda)_\mu$

Affine Grassmannian slices

We define affine Grassmannian slices.

- For $\mu \in P_+$, $\mathcal{W}_\mu = G_1^\vee[t^{-1}]t^\mu$
- If $\mu \leq \lambda$, $\overline{\mathcal{W}}_\mu^\lambda = \overline{Gr^\lambda} \cap \mathcal{W}_\mu$

Theorem

1. *$\overline{\mathcal{W}}_\mu^\lambda$ has a natural Poisson structure*
2. *$\overline{Gr^\lambda} \cap S^\mu \subset \overline{\mathcal{W}}_\mu^\lambda$ is a Lagrangian subvariety and is the attracting set for a Hamiltonian $\mathbb{C}^\times$ action.*

Truncated shifted Yangians

Braden-Licata-Proudfoot-Webster philosophy: if X is a conical symplectic singularity with Hamiltonian $\mathbb{C}^\times$-action, then we can categorify $H_{top}(X_+)$ using deformation quantization.

- Find $\mathcal{A}$, a filtered algebra, such that $gr(\mathcal{A}) \cong \mathbb{C}[X]$.
- The $\mathbb{C}^\times$-action on X gives a grading on $\mathcal{A}$, and we study category $\mathcal{O}$ for this grading.
- We have a cycle map $K(\mathcal{A}\text{-mod}^{\mathcal{O}}) \to H(X_+)$.

In a series of papers with Tingley, Webster, Weekes, and Yacobi, we studied the quantizations of $\overline{\mathcal{W}}^\lambda_\mu$, called truncated shifted Yangians Y^λ_μ. They are subquotients of the Yangian.

Theorem (KTWWY)

$$\bigoplus_\mu Y^\lambda_\mu\text{-mod}^{\mathcal{O}} \textit{ categorifies } V(\lambda)$$

Higgs and Coulomb branches

G a complex reductive group, V a representation of G

Physicists define a gauge theory from G, V and two spaces:

- Higgs branch

$$T^*V \;/\!/\!/\!/\; G = \mu^{-1}(0) \;/\!/\; G$$

- Coulomb branch

$$\text{roughly} \quad \operatorname{Spec} H_*(Maps(pt, V/G))$$

Example

$G = \mathbb{C}^\times, V = \mathbb{C}^n$

- Higgs branch

$$\{A \in M_n(\mathbb{C}) : A^2 = 0, \operatorname{rank} A \leq 1\}$$

- Coulomb branch

$$\mathbb{C}^2 \;/\!/\; \mathbb{Z}/n$$

Coulomb branch algebra

Definition due to Braverman-Finkelberg-Nakajima

- $\mathcal{K} = \mathbb{C}((t)),\ \mathcal{O} = \mathbb{C}[[t]],\ Gr_G := G(\mathcal{K})/G(\mathcal{O})$
- $T_{G,V} := \{([g], v) \in Gr_G \times V \otimes \mathcal{K} : v \in g(V \otimes \mathcal{O})\} \to Gr_G$, a vector bundle with fibre $V \otimes \mathcal{O}$
- $Z_{G,V} := T_{G,V} \times_{V \otimes \mathcal{K}} T_{G,V}$, Steinberg variety

$$Z_{G,V}/G(\mathcal{K}) = \{(P, \sigma) : P \text{ is a principal } G\text{-bundle on } D \sqcup_{D^\times} D$$
$$\sigma \text{ is a section of } P \times^G V\ \}$$

- $\mathcal{A}(G, V) = H_*^{G(\mathcal{K})}(Z_{G,V})$, commutative convolution algebra
- $\mathcal{A}_\hbar(G, V) = H_*^{G(\mathcal{K}) \rtimes \mathbb{C}^\times}(Z_{G,V})$, non-commutative deformation

$\operatorname{Spec} \mathcal{A}(G, V)$ is the **Coulomb branch**, $\mathcal{A}_\hbar(G, V)$ is the **Coulomb branch algebra**

Quiver gauge theories

Let $\lambda \in P_+, \mu \in P$, with $\mu \leq \lambda$.
Write $\lambda - \mu = \sum_i v_i \alpha_i$ and $\lambda = \sum_i w_i \omega_i$

$$V = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_i \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

where the first sum ranges over edges in the Dynkin quiver.

$$\mathcal{M}_\mu^\lambda := T^* V /\!\!/\!\!/\!\!/_\chi G$$

is a Nakajima quiver variety, where $G = \prod GL_{v_i}$.

Theorem (Braverman-Finkelberg-Nakajima+KKWW)

Assume μ is dominant.

1. $\mathrm{Spec}\mathcal{A}(G, V) = \overline{\mathcal{W}}_\mu^\lambda$
2. $\mathcal{A}_\hbar(G, V) = Y_\mu^\lambda$

Generalized affine Grassmannian slices

Using Coulomb branches, we can generalize the definition of affine Grassmannian slice and truncated shifted Yangians to non-dominant μ.

$$\mathcal{W}_\mu = N^\vee_-[t^{-1}]t^\mu N^\vee[t] \qquad \overline{\mathcal{W}}^\lambda_\mu = \mathcal{W}_\mu \cap \overline{G^\vee[[t]]t^\mu G^\vee[[t]]}$$

and Y^λ_μ can be defined by generators and relations, though it is no longer a Yangian subquotient.

Theorem (Krylov)

$\overline{\mathcal{W}}^\lambda_\mu$ contains $\overline{Gr^\lambda} \cap S^\mu$ as a Lagrangian attracting set.

An equivalence

In order to establish our categorification,

Theorem

$$\bigoplus_\mu Y_\mu^\lambda\text{-mod}^{\mathcal{O}} \textit{ categorifies } V(\lambda)$$

we related these categories to modules for KLR algebras.
Let T_μ^λ denote a KLRW-algebra (has w_i red strands labelled i and v_i black strands labelled i). Let $_{-}T_\mu^\lambda$ be the quotient by the diagrams where the leftmost strand is black. From Webster's work, modules for this algebra categorifies $V(\lambda)_\mu$ (or tensor products).

Theorem (KTWWY)

$$_{-}T_\mu^\lambda\text{-mod} \cong Y_\mu^\lambda\text{-mod}^{\mathcal{O}}$$

Mysterious functors

From the categorical $\mathfrak{g}$-action on the KLRW algebra modules, we have functors

$$E_i : \ \_T^\lambda_\mu\text{-mod} \to \ \_T^\lambda_{\mu+\alpha_i}\text{-mod}$$

By *transport de structure*, we obtain a functors

$$E_i : Y^\lambda_\mu\text{-mod}^{\mathcal{O}} \to Y^\lambda_{\mu+\alpha_i}\text{-mod}^{\mathcal{O}}$$

How can we describe these functors in Coulomb branch terms? We want to relate

$$\mathcal{A}(G, V) \text{ and } \mathcal{A}(G', V')$$

where G, V comes from a quiver and G', V' the corresponding quiver where we reduce the ith vertex by 1.

Parabolic restriction for Coulomb branch algebras

Let G, V be arbitrary and let $\xi : \mathbb{C}^\times \to G$ be a coweight. Let L be the centralizer of ξ, P the corresponding parabolic.

Theorem

The algebras $\mathcal{A}_\hbar(G, V)$ and $\mathcal{A}_\hbar(L, V^\xi)$ are related by

1. *A Morita equivalence between $\mathcal{A}_\hbar(G, V)$ and $\mathcal{A}_\hbar^P(G, V)$*
2. *An inclusion of algebras $\mathcal{A}_\hbar(L, V) \hookrightarrow \mathcal{A}_\hbar^P(G, V)$*
3. *An isomorphism $\mathcal{A}_\hbar(L, V)[r_\xi^{-1}] \cong \mathcal{A}_\hbar(L, V^\xi)$*

This theorem allows us to construct a nice restriction functor

$$res_\xi : \mathcal{A}_\hbar(G, V)\text{-mod} \to \mathcal{A}_\hbar(L, V^\xi)\text{-mod}$$

which preserves categories of weight modules.

Return to the quiver case

Fix λ, μ as before and let G, V be the resulting quiver theory. Write $\lambda - \mu = \sum_i v_i \alpha_i$ and $\lambda = \sum_i w_i \omega_i$.

$$G = \prod_i GL_{v_i} \quad V = \bigoplus_{i \to j} \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{v_j}) \oplus \bigoplus_i \mathrm{Hom}(\mathbb{C}^{v_i}, \mathbb{C}^{w_i})$$

Take $\xi : \mathbb{C}^\times \xrightarrow{\omega_1} GL_{v_i} \hookrightarrow G$.
So $\mathcal{A}_\hbar(G, V) = Y^\lambda_\mu$ and $\mathcal{A}_\hbar(L, V^\xi) = Y^\lambda_{\mu+\alpha_i} \otimes D(\mathbb{C}^\times)$.

Theorem

The resulting functor $res_\xi : Y^\lambda_\mu\text{-mod}^\mathcal{O} \to Y^\lambda_{\mu+\alpha_i}\text{-mod}^\mathcal{O}$ *agrees with the functor* E_i *defined using the equivalence with KLRW-algebra modules.*